

Uppgifter 7

TAOP07 – Optimeringslära grundkurs

Skriven av Oliver Wettergren

oliwe188@student.liu.se

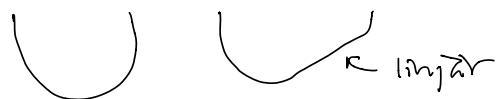
<https://www.instagram.com/olwettergren/>

Lektionsgenomgång

$g(x)$ convex $\Rightarrow \underline{X} = \{x : g \leq b\}$ convex
 concave \geq

$f(x)$ convex $\Leftrightarrow H(x) \geq 0 \quad \forall x \in \underline{X}$
 on \underline{X}

$f(x)$ is strictly convex on $\underline{X} \Leftrightarrow H(x) > 0 \quad \forall x \in \underline{X}$



$$H \stackrel{<}{{}_\sim} 0 \Leftrightarrow \lambda_i \stackrel{<}{{}_\sim} 0, \quad \forall i$$

$$H > 0 \Leftrightarrow \lambda_i > 0, \quad \forall i$$

$$H > 0 \Leftrightarrow \det h_1 > 0$$

$$\det h_2 > 0$$

$$\det h_3 > 0$$

$$\det h_4 > 0$$

$$H = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

h_1 h_4

$$H < 0 \Leftrightarrow \det h_1 < 0$$

$$\det h_2 > 0$$

$$\det h_3 < 0$$

$$\det h_4 > 0$$

Optimality conditions $\min_{x \in \mathbb{R}^n} f(x)$

x^* is local min $\Rightarrow \nabla f(x^*) = 0$
 $H(x^*) \geq 0$

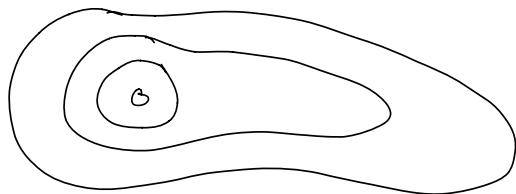
$$x^* \text{ is strict local min} \Leftrightarrow \nabla f(x^*) = 0 \\ H(x^*) > 0$$

If $f(x)$ is convex then:

$$x^* \text{ is global min} \Leftrightarrow \nabla f(x^*) = 0$$

Levelset

$$L(c) = \{x : f(x) \leq c\}$$



$$x_{k+1} = x_k + t d_k$$

\uparrow \uparrow
 $t \in \mathbb{R}^n$ search direction
 $\in \mathbb{R}$ step-length

$$f(x_k + t d_k) = \min_{t \geq 0} f(x_k + t d_k)$$

Def:

$d \in \mathbb{R}^n$ is a descent direction for $f(x)$ in x if
 $d^\top \nabla f(x) < 0$

If d is a descent direction then $f(\bar{x} + t d) < f(\bar{x})$
 & sufficiently small $t > 0$.

Steepest descent method

$$d_k = - \underbrace{\nabla f(x_k)}_{\nabla f_k}$$

Newton's method

$$d_k = - \underbrace{H^{-1}(x_k)}_{H_k} \nabla f(x)$$

$$H_k d_k = - \nabla f_k$$

$$m_k(x) = f_k + \nabla f_k^T (x - x_k) + \frac{1}{2} (x - x_k)^T H_k (x - x_k)$$

Assumption:

$$H_k > 0 \Rightarrow m_k(x) \text{ is convex}$$

$$\nabla m_k(\bar{x}) = 0 = \nabla f_k + M_k(\bar{x} - x_k)$$

Newton's method \Rightarrow

$$x^2, H > 0, \text{ convex-strictly}$$

Ex:

$$f(x) = (x_1 + x_2)^2 = x_1^2 + 2x_1 x_2 + x_2^2$$

a) Show that it is a convex function

b) Find the Newton search direction

$$H(x) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 4 \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 4 \end{cases}$$

$$\Rightarrow H \geq 0$$

$$b) d_u = -H^{-1}(x_u) \nabla f(x_u) = -\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2(x_1+x_2) \\ 2(x_1+x_2) \end{pmatrix} = -\begin{pmatrix} 16 \\ 16 \end{pmatrix} = -16 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

H^{-1} does not exist

$$\left| \begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array} \right|$$

Ex: Consider

$$\min f(x) = 4(x_1^2 + x_2^2) - 2x_1 - x_1^2 x_2$$

a) Use the steepest descent method with

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Stop when $\|\nabla f(x)\| < 0,01$

$$\nabla f(x) = \begin{pmatrix} 8x_1 - 2x_1 x_2 - 2 \\ 8x_2 - x_1^2 \end{pmatrix}$$

$k=0$

$$f(x_0) = 0$$

$$\nabla f_0 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad \|\nabla f_0\| = \sqrt{(-2)^2 + 0^2} = 2 > 0,01 \Rightarrow$$

$$\text{continue, } d_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = -\nabla f_0$$

$$x_1(t) = x_0 + t d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \end{pmatrix}, \quad \varphi(t) = f(x_0 + t d_0) = 16t^2 - 4t, \quad \varphi'' = 32 > 0 \Rightarrow \varphi(t) \text{ convex}$$

$$\Rightarrow \varphi'(t) = 0, \quad 32t - 4 = 0 \Rightarrow t_0 = \frac{1}{8} \Rightarrow x_1 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix},$$

$$f(x_1) = -\frac{1}{4} < f(x_0) \text{ ok!}$$

k=1

$$\nabla f_1(x_1) = \begin{pmatrix} 0 \\ 1/16 \end{pmatrix}, \quad \|\nabla f_1\| = \sqrt{0^2 + (\frac{1}{16})^2} = \frac{1}{16} > 0, 01,$$

$$d_1 = -\begin{pmatrix} 0 \\ 1/16 \end{pmatrix},$$

$$x_2(t) = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1/16 \end{pmatrix} = \begin{pmatrix} 1/4 \\ t/16 \end{pmatrix}$$

$$\psi(t) = 4 \left(\frac{1}{16} + \frac{t^2}{16^2} \right) - 2 \cdot \frac{1}{4} - \frac{1}{16} \cdot \frac{t}{16}, \quad \psi' = \frac{8}{16^2} \geq 0 \Rightarrow$$

$$\text{convex } \psi(t) = \psi'(t) = 0, \quad \psi''(t) = \frac{4 \cdot 2 \cdot t}{16^2} - \frac{1}{16^2} = 0 \\ \Rightarrow t_1 = \frac{1}{8} \Rightarrow$$

$$x_2 = \begin{pmatrix} 1/4 \\ 1/128 \end{pmatrix}, \quad f(x_2) = -\frac{256}{128} \approx -1,98 < -0,25 = f(x_1) \text{ ok!}$$

k=2

$$\nabla f_2 = \begin{pmatrix} -1/256 \\ 0 \end{pmatrix}, \quad \|\nabla f_2\| \leq \frac{1}{100} \Rightarrow \text{stop.}$$

b) Can we hope that x_2 is close to a local min.

$$H(x) = \begin{pmatrix} 8 - 2x_2 & -2x_1 \\ -2x_1 & 8 \end{pmatrix} \quad \begin{aligned} \det h_{11} &= 8 - 2x_2 > 0 \\ \det h_{22} &= 4x_2 + x_1^2 > 0 \end{aligned} \quad \left. \begin{aligned} H(x) &> 0 \\ \|\nabla f_2\| &\approx 0 \end{aligned} \right\} \Rightarrow$$

\Rightarrow Yes, we may say that x_2 is close to local min.

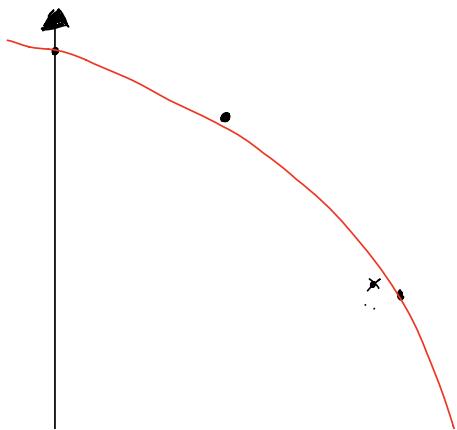
(118)

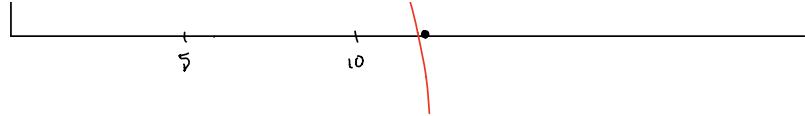
Kund	x-koordinat	y-koordinat	Antal sändningar
1	5	10	200
2	10	5	150
3	0	12	200
4	12	0	300

a)

$$\min z = 200 \sqrt{(x-5)^2 + (y-10)^2} + 150 \sqrt{(x-10)^2 + (y-5)^2} +$$

$$+ 200 \sqrt{(x^2 + (y-12)^2)} + 300 \sqrt{(x-12)^2 + y^2}$$





JÄ, tS problemet är leverest.

$$\begin{aligned} f'_{x_1} &= 4(x_1 - 3) - x_2 & f''_{x_1 x_1} &= 4 \\ f'_{x_2} &= 2(x_2 - 2) - 3 - x_1 & f''_{x_2 x_2} &= 2 \end{aligned}$$

$$f''_{x_1 x_2} = -1$$

$$\Rightarrow \begin{vmatrix} (4-\lambda) & -1 \\ -1 & (2-\lambda) \end{vmatrix} = (4-\lambda)(2-\lambda) - 1 = 0$$

$$\Leftrightarrow 8 - 6\lambda + \lambda^2 - 1 = 0 \Leftrightarrow (\lambda - 3)^2 - 9 + 7 = 0$$

$$\Leftrightarrow \lambda = 3 \pm \sqrt{2}$$

$$f'_{x_1} = 3x_1^2 - 2x_1$$

$$f'_{x_2} = 6x_2^2 - 6$$

$$f''_{x_1 x_1} = 6x_1$$

$$f''_{x_1 x_2} = 0$$

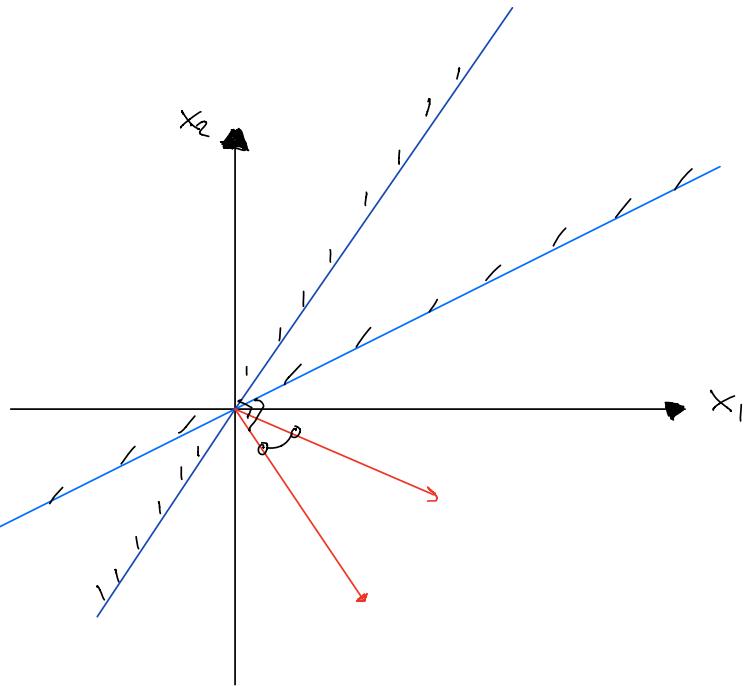
$$f''_{x_2 x_2} = 12x_2$$

$$\min f(x_1, x_2) = x_1$$

$$\text{då } (x_1 - 1)^2 + (x_2 + 2)^2 \leq 16$$

$$x_1^2 + x_2^2 \geq 13$$

(133)



$$\frac{x_1}{2} \geq x_2 \quad \text{och} \quad 3x_1 \geq x_2$$

$$\nabla f(x_1, x_2) = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \alpha, \beta \geq 0$$

(135)

$$\min f(x) = 2(x_1 - 5)^2 + (x_2 - 8)^2$$

$$\text{d.h.} \quad x_1^2 + x_2^2 \leq 25$$

$$x_1 + 4x_2 \leq 19$$

$$x_1, x_2 \geq 0$$

$$(34)^T$$

$$g_1(x_1, x_2) = x_1^2 + x_2^2 - 25 \Rightarrow \nabla g_1 = 2x_1 + 2x_2 \Rightarrow$$

$$g_2(x_1, x_2) = x_1 + 4x_2 - 19 \Rightarrow \nabla g_2 = (1+4) = 5$$

$$\Rightarrow \nabla f = \alpha \begin{pmatrix} 6 \\ 8 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \alpha, \beta \geq 0$$

$$\nabla f = \begin{pmatrix} 4(x_1 - 5), & 2(x_2 - 8) \end{pmatrix}^{(34)} = \begin{pmatrix} -8 \\ -16 \end{pmatrix}$$

a)

b)

c) Inga lokala min!

(165)

$$\min f(x_1, x_2) = (3x_1 - x_2)^2 + 4x_2^2 - x_1$$

$$\text{dci} \quad x_1^3 + x_2^2 \leq 37$$

$$-4x_1 + x_2 = 2$$

$$x_1, x_2 \neq 0$$

$$\nabla f = (6(3x_1 - x_2) - 1, -2(3x_1 - x_2) + 8x_2)$$

$$\begin{aligned} \nabla f(x^*) &= b \cdot (-3) - 1, -2 \cdot (-\cancel{b}) + 8b \rangle = \\ &= (-19, 54) \end{aligned}$$

$$\nabla g_1 = (3x_1^2, 2x_2)$$

$$\nabla g_2 = (-4, 1)$$

$$y_1 g_1(1, b) = y_1 (1 + 3b) - 37 = 0 \Rightarrow \text{Inget}$$

$$y_2 g_2(1, b) = y_2 (-4 + b) - 2 = 0 \Rightarrow \text{ger inget}$$

$$\nabla f(x^*) + \sum_{i=1}^m y_i^* \nabla g_i(x^*) = 0 \quad \text{ej uppfyllt}$$

$$\text{da } \nabla f(x^*) \neq 0$$

|

163

$$\text{mm } f(x_1, x_2) = \frac{1}{2} x_1^2 - 10x_1 x_2 + 10x_2^2$$

$$\begin{array}{l} 2x_1 + x_2^2 \leq 5 \\ x_1^2 - ax_2 \leq 2 \end{array}$$

$$\nabla f = (x_1 - 10x_2, -10x_1 + 20x_2) \Rightarrow x^* = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

$$\nabla f(2,1) = (-8, 0)$$

$$g_1(x) = 2x_1 + x_2^2 - 5 \leq 0 \Rightarrow g_1(x^*) = 0 \Rightarrow \text{ger mjet}$$

$$g_2(x_1, x_2) = x_1^2 - ax_2 - 2 \leq 0 \quad g_2(x^*) = 4 - a - 2 \leq 0$$

$$\nabla g_1 = (2, 2x_2) = (2, 2) \Rightarrow \underline{\underline{a \geq 2}}$$

$$\nabla g_2 = (2x_1, -a) = (4, -a)$$

$$\bullet y_2^* g_2(x^*) = \underline{\underline{y_2^*(2-a) = 0}}$$

$$(1) \Rightarrow (-8, 0) + y_1(2, 2) + y_2(4, -a) = 0$$

$$\Rightarrow \bullet \begin{pmatrix} -8 + 2y_1 + 4y_2 \\ 0 + 2y_1 - ay_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} 2y_1 + 4y_2 = 8 \\ 2y_1 - ay_2 = 0 \end{cases} \sim \begin{cases} 2y_1 + 4y_2 = 8 \\ -y_2(4+a) = -8 \end{cases} \quad (2)$$

$$\begin{aligned}
 & \text{Given: } \underline{\underline{y_2(2-a) = 0}} \quad | \quad y_2(2-a) = 0 \quad (3) \quad \frac{(3)}{(2)} \\
 \sim & \left\{ \begin{array}{l} 2y_1 + 4y_2 = 8 \\ -y_2(4+a) = -8 \\ -\frac{2+a}{4+a} = 0 \Rightarrow a=2 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 2y_1 + 4y_2 = 8 \\ -6y_2 = -8 \end{array} \right. \\
 \Rightarrow & y_1 = \frac{8 - 4 \cdot \frac{8}{6}}{2} = \frac{8 - \frac{16}{3}}{2} = \frac{24 - 16}{6} = \frac{8}{3} = \frac{4}{3} \geq 0 \\
 & y_2 = \frac{8}{6} = 0
 \end{aligned}$$

Så $a=2$, KKT-villkoren uppfylls

(168)

$$\min e^{x_1^2 - 2x_1} - \ln(7x_2 + 1)$$

$$\text{då } x_1 + x_2 \geq 5$$

$$x_1^2 + x_2^2 + 2x_2 \leq 40$$

$$x_2 \geq 0$$

- a) Undersök om problemet är konvext, dvs mälfunktionen konvex och bivillkoren utgör en konkav mängd.
- b) Om problemet är givet konvext är optimum unikt.