

# Lösningsgång

TAOP07 – Optimeringslära grundkurs

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Skriven av Oliver Wettergren

[oliwe188@student.liu.se](mailto:oliwe188@student.liu.se)

<https://www.instagram.com/olwettergren/>

① T tidsperioder

Kostnad  $f_t$  (inköp) och  $c_t$  (rörlig)

Max inköp  $u_t$

Åtgången  $d_t$

Lagerkostnad  $h_t$

Maxkapacitet  $L$

Start:  $I_0$  i lager

Slut  $I_T$  i lager

$$I_0 + \text{köp} - \text{solj} = I_T$$

•  $x_t =$  mängd enheter som ska köpas in i period  $t$ ,  
 $t = 1, \dots, T$

•  $y_t = \begin{cases} 1, & \text{om } x \geq 0 \\ 0 & \text{annars} \end{cases} \quad t = 1, \dots, T$

•  $L_t =$  mängd enheter som lagerväcks i period  $t$ ,  
 $t = 1, \dots, T$

$$\min z = \sum_{t=1}^T (f_t y_t + x_t \cdot c_t + h_t L_t)$$

då

$$x_t \leq u_t y_t$$

$$L_t = L_{t-1} + x_t - d_t$$

$$L_0 = I_0$$

$$L_t = I_T$$

$$L_t \leq L$$

$$x_t, L_t \geq 0, \quad u = 0/1$$

$$\textcircled{2} \quad z^* = \min z = 30x_1 + 25x_2 + 29x_3 + 8x_4 + 16x_5$$

då

$$6x_1 + 7x_2 + 8x_3 + 2x_4 + 3x_5 = 13$$

$$5x_1 + 2x_2 + x_3 + x_4 + 3x_5 = 5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 5 \end{pmatrix} = \frac{1}{7-4} \begin{pmatrix} 1 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 13 \\ 5 \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \geq \bar{0} \Rightarrow \text{baslösning tillåten}$$

Komplementär baslösning

$$y^T = c_B^T B^{-1} = \left( (25 \ 8) \frac{1}{3} \begin{pmatrix} 1 & -2 \\ -2 & 7 \end{pmatrix} \right)^T = \frac{1}{3} \begin{pmatrix} 9 \\ 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}}$$

två fria.

$$\bar{c}_1 = c_1 - y^T A_1 = 30 - (3 \ 2) \begin{pmatrix} 6 \\ 5 \end{pmatrix} = 30 - 28 = 2 \geq 0$$

$$\bar{c}_3 = c_3 - y^T A_3 = 29 - (3 \ 2) \begin{pmatrix} 8 \\ 1 \end{pmatrix} = 29 - 26 = 3 \geq 0$$

$$\bar{c}_5 = c_5 - y^T A_5 = 16 - (3 \ 2) \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 16 - (9+6) = 1 \geq 0$$

Alla icke-negativa  $\Rightarrow$  baslösningen är optimal.

b) Reducerade kostnader för baserna:

$$\bar{c}_2 = c_2 - y^T A_2 = 25 - (3, 2) \begin{pmatrix} 7 \\ 2 \end{pmatrix} = 25 - 25 = 0$$

$$\bar{c}_4 = c_4 - y^T A_4 = 8 - (3, 2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 8 - 8 = 0$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 13-\delta \\ 5-\delta \end{pmatrix} = \frac{1}{7-4} \begin{pmatrix} 1 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 13-\delta \\ 5-\delta \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 13-\delta-10+2\delta \\ -26+2\delta+35-7\delta \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3+\delta \\ 9-5\delta \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \delta \\ -5\delta \end{pmatrix}$$

$$\Rightarrow 1 + \frac{1}{3} \delta \geq 0 \Leftrightarrow \delta \geq -3$$

$$3 - \frac{1}{3} 5\delta \geq 0 \Leftrightarrow \frac{1}{3} 5\delta \leq 3 \Leftrightarrow \delta \leq \frac{9}{5}$$

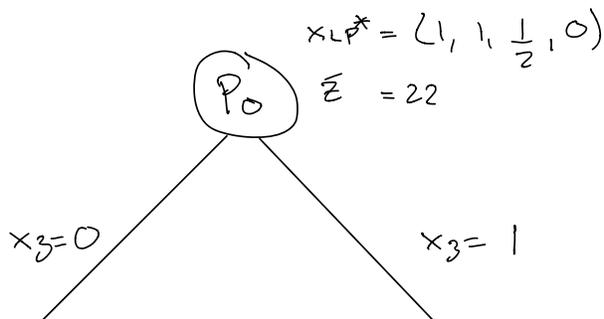
$\left. \begin{matrix} (-3 \leq) \\ \text{men } \delta \end{matrix} \right\} \Rightarrow \underline{\underline{\delta \leq \frac{9}{5}}}$

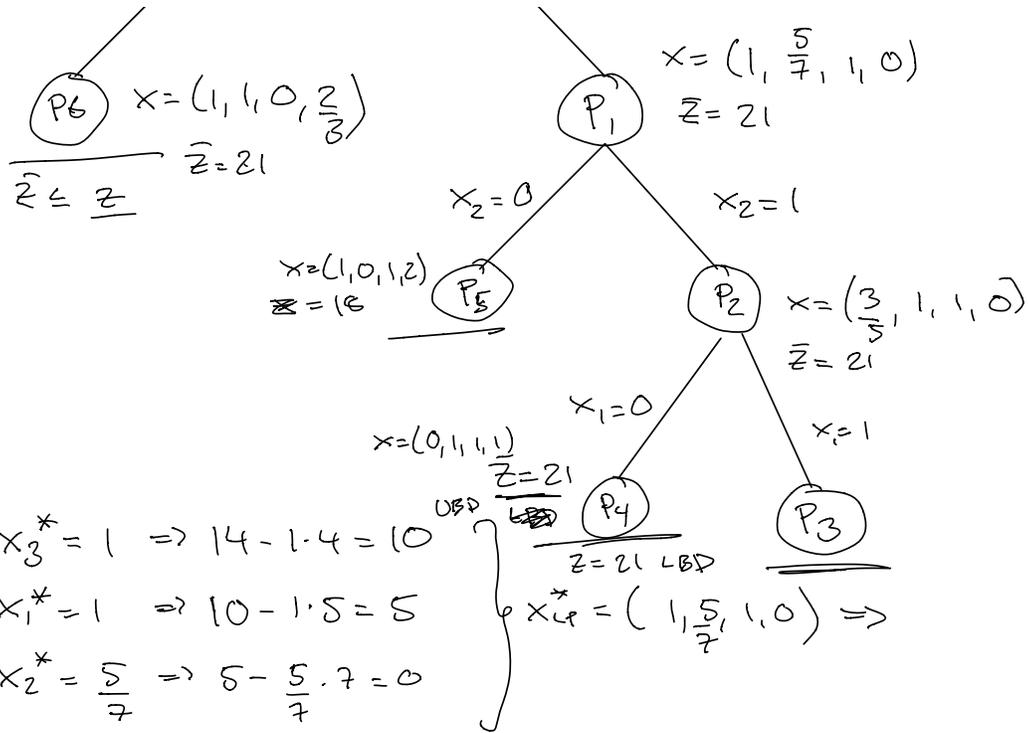
③  $\max z = 8x_1 + 11x_2 + 6x_3 + 4x_4$   
 $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$   
 $x_1, x_2, x_3, x_4 = 0/1$

a)  $\frac{8}{5} > \frac{11}{7} > \frac{6}{4} > \frac{4}{3} \Rightarrow$  använd variablerna  
i ordningen  $x_1, x_2, x_3, x_4$

$$\left. \begin{array}{ll} x_1^* = 1 & 14 - 1 \cdot 5 = 9 \\ x_2^* = 1 & 9 - 1 \cdot 7 = 2 \\ x_3^* = 1/2 & 2 - \frac{1}{2} \cdot 4 = 0 \\ x_4^* = 0 & 0 \end{array} \right\} \Rightarrow x_{LP}^* = \left( 1, 1, \frac{1}{2}, 0 \right)$$

$$\Rightarrow z_{LP}^* = 22 \Rightarrow z^* \leq \bar{z} = 22$$





$P_1$ :

$$\begin{aligned}
 x_3^* = 1 &\Rightarrow 14 - 1 \cdot 4 = 10 \\
 x_1^* = 1 &\Rightarrow 10 - 1 \cdot 5 = 5 \\
 x_2^* = \frac{5}{7} &\Rightarrow 5 - \frac{5}{7} \cdot 7 = 0
 \end{aligned}$$

$$x_4^* = 0$$

$x_{LP}^* = (1, \frac{5}{7}, 1, 0) \Rightarrow$

$$\begin{aligned}
 z_{LP}^* &= 8 + \frac{11 \cdot 5}{7} + 6 = \frac{56}{7} + \frac{55}{7} + \frac{42}{7} = \\
 &= \frac{111}{7} + \frac{42}{7} = \left[ \frac{153}{7} \right] = \underline{\underline{21}}
 \end{aligned}$$

$P_2$ :

$$\begin{aligned}
 x_3^* = 1 &\Rightarrow 14 - 1 \cdot 4 = 10 \\
 x_2^* = 1 &\Rightarrow 10 - 7 = 3 \\
 x_1^* = \frac{3}{5} &\Rightarrow 3 - \frac{3}{5} \cdot 5 = 0 \\
 x_4^* &= 0
 \end{aligned}$$

$$\Rightarrow x_{LP}^* = (\frac{3}{5}, 1, 1, 0)$$

$$z_{LP}^* = 8 \cdot \frac{3}{5} + 11 + 6 = \frac{24}{5} + \frac{55}{5} + \frac{30}{5} = \left[ \frac{109}{5} \right] = 21$$

$P_4$ :

$$x_3^* = 1 \Rightarrow 10$$

$$x_2^* = 1 \Rightarrow 3$$

$$x_4^* = 1 \Rightarrow 3 \cdot 1 - 3 = 0$$

$$x_1^* = 0$$

$$x_{LP}^* = (0, 1, 1, 1)$$

$$\underline{\underline{z = 21}}$$

P5:

$$x_3^* = 1 \Rightarrow 10$$

$$x_2^* = 0$$

$$x_1^* = 1 \Rightarrow 10 - 4 = 6$$

$$x_4^* = 2 \Rightarrow 6 - 1 \cdot 3 = 3$$

$$\left. \begin{array}{l} x_3^* = 1 \Rightarrow 10 \\ x_2^* = 0 \\ x_1^* = 1 \Rightarrow 10 - 4 = 6 \\ x_4^* = 2 \Rightarrow 6 - 1 \cdot 3 = 3 \end{array} \right\} \Rightarrow x_{LP}^* = (1, 0, 1, 1)$$

$$\Rightarrow \underline{z = 18}$$

P6

$$x_3^* = 0$$

$$x_1^* = 1 \Rightarrow 14 - 1 \cdot 5 = 9$$

$$x_2^* = 1 \Rightarrow 9 - 1 \cdot 7 = 2$$

$$x_4^* = \frac{2}{3} \Rightarrow 2 - \frac{2 \cdot 3}{3}$$

$$\left. \begin{array}{l} x_3^* = 0 \\ x_1^* = 1 \Rightarrow 14 - 1 \cdot 5 = 9 \\ x_2^* = 1 \Rightarrow 9 - 1 \cdot 7 = 2 \\ x_4^* = \frac{2}{3} \Rightarrow 2 - \frac{2 \cdot 3}{3} \end{array} \right\} \Rightarrow x_{LP}^* = (1, 1, 0, \frac{2}{3})$$

$$z^* = [ \dots ] = 21$$

Svar:  $z^* = 21$  då  $x^* = (1, 1, 1, 1)$

b) Undersök  $(1, 1, \frac{1}{2}, 0)$

(i)  $x_1 + x_3 = 1 + \frac{1}{2} \neq 1 \Rightarrow$  ja

(ii)  $x_1 + x_2 + x_3 = 1 + 1 + \frac{1}{2} \neq 2 \Rightarrow$  ja

(iii)  $x_1 + x_2 + x_3 + x_4 = 1 + 1 + \frac{1}{2} + 0 \leq 3 \Rightarrow$  nej

(i)  $x = (1, 0, 1, 0)$  tillåten lösning

men uppfyller inte  $x_1 + x_3 \leq 1 \Rightarrow$  inte giltig

(ii)  $x = (1, 1, 1, 0)$  ej tillåten lösning vilket

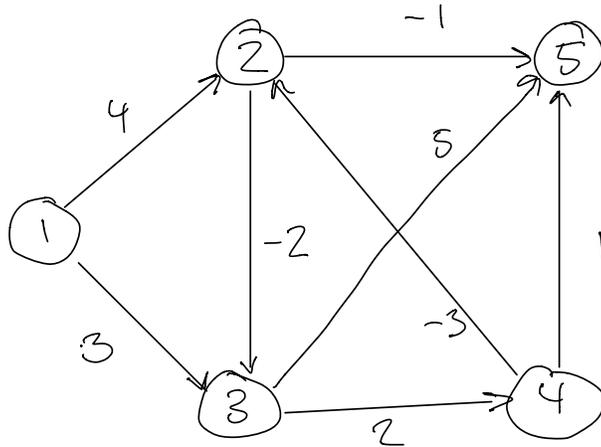
ger att  $x_1^* + x_2^* + x_3^* \leq 2$  giltig  $\Rightarrow$  giltig

(iii)  $x = (1, 1, 1, 1)$  ej tillåten lösning  $\Rightarrow$

$x_1 + x_2 + x_3 + x_4$  måste gälla  $\Rightarrow$  giltig.

⇒ endast (ii) giltig.

④



a) Formulering:

Variabler:

$$x_{ij} = \begin{cases} 1 & \text{om bägge } (i,j) \text{ ingår i vägen} \\ 0 & \text{annars} \end{cases}$$

$$\min z = 4x_{12} + 3x_{13} - 2x_{23} - x_{25} + 5x_{35} + 2x_{34} - 3x_{42} + x_{45}$$

då  $x_{12} + x_{13} = 1$

$$x_{23} + x_{25} = x_{12} + x_{42}$$

$$x_{34} + x_{35} = x_{13} + x_{23}$$

$$x_{34} = x_{42} + x_{45}$$

$$x_{45} + x_{35} + x_{25} = 1$$

$$x_{ij} = 0/1 \quad \forall i,j$$

$\Leftrightarrow \min z = (1, 3, 1, 4, 1, 2) \cdot x$   
 då  $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$  med  $\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$   
 bägge: (1,2) (1,3) (2,3) (2,4) (1,2)(3,4)  
 $1 \leq x_{ij} \leq 0$  och heltal,  $\forall i,j$

$$\begin{array}{rcccccc|l}
\Rightarrow & x_{12} & x_{13} & & & & = 1 & \gamma_1 \\
& -x_{12} & & x_{23} - x_{25} & & & -x_{42} & = 0 & \gamma_2 \\
& & -x_{13} - x_{23} & & +x_{34} + x_{35} & & & = 0 & \gamma_3 \\
& & & & -x_{34} & & x_{42} + x_{45} & = 0 & \gamma_4 \\
& & & -x_{25} & & -x_{35} & & -x_{45} & = -1 & \gamma_5
\end{array}$$

$$\begin{aligned}
\Rightarrow \max w &= y_1 - y_5 \\
\text{d.h.} \quad y_1 - y_2 &\leq 4 \\
y_1 - y_3 &\leq 3 \\
y_2 - y_3 &\leq -2 \quad (1) \\
y_2 - y_5 &\leq -1 \\
y_3 - y_4 &\leq 2 \quad (2) \\
y_3 - y_5 &\leq 5 \\
-y_2 + y_4 &\leq -3 \quad (3) \\
y_4 - y_5 &\leq 1
\end{aligned}$$

$$\left. \begin{array}{l}
y_2 \leq -2 + y_3 \\
y_3 \leq 2 + y_4 \\
y_2 \geq y_4 + 3
\end{array} \right\} \Rightarrow \begin{aligned}
&y_4 + 3 \leq -2 + y_3 \\
&\Rightarrow y_4 + 5 \leq y_3 \\
&\Rightarrow y_4 + 5 \leq 2 + y_4 \\
&\Rightarrow \underline{\underline{0 \leq -3}}
\end{aligned}$$

$\Rightarrow$  Duala villkoren motstridiga  $\Rightarrow$   
tillåten lösning saknas

Primala problemet har tillåten lösning  
 $\Rightarrow$  obegränsat optimum.

⑤

$$\max x_1, x_2$$

$$\text{då } x_1 + x_2^2 \leq 2$$

$$x_1, x_2 \geq 0$$

$$a) \bar{x} = \left( \frac{4}{5}, \sqrt{\frac{2}{3}} \right)^T$$

$$\nabla f(x) = (x_2, x_1)$$

$$\nabla g(x) = (1, 2x_2)$$

$$\nabla f(\bar{x}) = v_i \nabla g_i(x) \quad v_i \geq 0$$

$$g_i(x) = b_i$$

$$\nabla f(\bar{x}) = \left( \frac{\sqrt{2}}{3}, \frac{4}{5} \right)$$

$$v_i (b_i - g_i(x)) = 0$$

$$\nabla g(\bar{x}) = \left( 1, 2\sqrt{\frac{2}{3}} \right)$$

$$\nabla f(\bar{x}) = \nabla v_i g_i(\bar{x}) \Rightarrow \begin{pmatrix} \sqrt{2}/3 \\ 4/5 \end{pmatrix} = v_i \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}$$

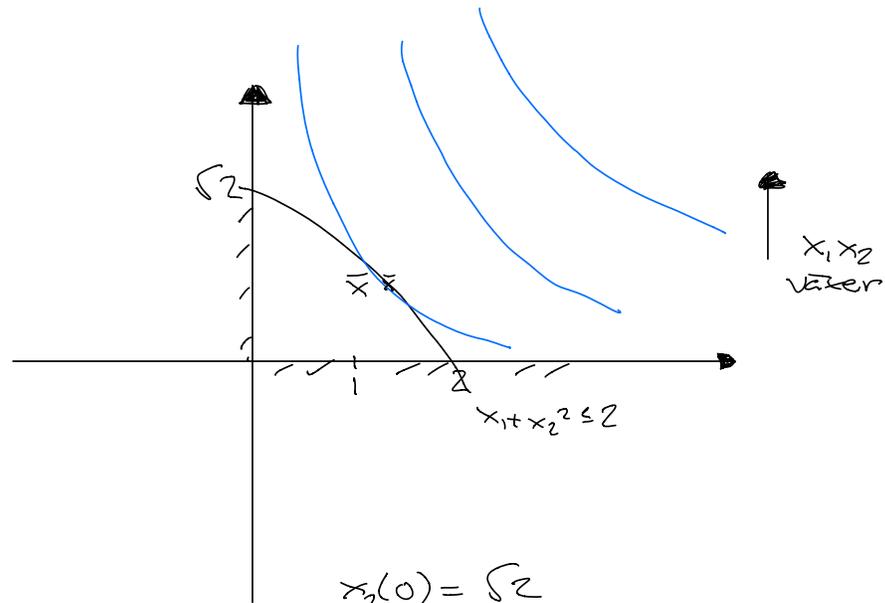
$$\Leftrightarrow \begin{pmatrix} \sqrt{2}/3 \\ 4/5 \end{pmatrix} = \begin{pmatrix} v_i \\ 2\sqrt{2} v_i \end{pmatrix}$$

$$\Rightarrow v_i = \frac{\sqrt{2}}{3} \Rightarrow \begin{pmatrix} \sqrt{2}/3 \\ 2 \cdot \frac{\sqrt{2}}{3} \cdot \frac{\sqrt{2}}{3} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/3 \\ 4/3 \end{pmatrix}$$

$$v_i = \frac{\sqrt{2}}{3} \geq 0$$

$$v_i (b_i - g(x)) = 0 \quad \underline{\underline{\text{KKT}}}$$

b)



$$x_2(0) = \sqrt{2}$$

$$x_2(1) = \sqrt{2-1} = \sqrt{1} = 1$$

$$x_2(2) = 0$$

$$x_1, x_2 = u, \quad u \neq 0 \Rightarrow x_2 = \frac{u}{x_1}$$

⑥  $z^* = \min \quad 5x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 8x_6$   
 då  $x_1 + x_2 + x_5 + x_6 \geq 1$   
 $x_1 + x_4 \geq 1$   
 $x_4 + x_6 \geq 1$   
 $x_3 + x_5 + x_6 \geq 1$   
 $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$   
 $x_1, \dots, x_6 = 0/1$

$$v = (v_1, v_2, v_3, v_4) \geq 0$$

$$\Rightarrow h(v) = \min 5x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 8x_6 +$$

$$\textcircled{A} + v_1(1 - x_1 - x_2 - x_5 - x_6) + v_2(1 - x_1 - x_4) + v_3(1 - x_4 - x_6) +$$

$$+ v_4(1 - x_3 - x_5 - x_6)$$

$$\text{då } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$$

$$x_1, \dots, x_6 = 0/1$$

$$\Rightarrow \min v_1 + v_2 + v_3 + v_4 + (5 - v_1 - v_2)x_1 +$$

$$+ (3 - v_1)x_2 + (4 - v_4)x_3 + (6 - v_2 - v_3)x_4 +$$

$$+ (7 - v_1 - v_4)x_5 + (8 - v_1 - v_3 - v_4)x_6$$

$$\text{då } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$$

$$x_1, \dots, x_6 = 0/1$$

$$h^* = \max h(v)$$

$$v \geq 0$$

$$v(1, 1, 1, 2) = 5 + \min 3x_1 + 2x_2 + 2x_3 + 4x_4 +$$

$$+ 4x_5 + 4x_6$$

$$\text{då } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$$

$$x_1, \dots, x_6 = 0/1$$

$$= 5 + 2 + 2 = 9 \text{ för } v = (1, 1, 1, 2) = (0, 1, 1, 0, 0, 0)$$

Tillåten lösning?

$$x_1 + x_4 \neq 1 \text{ och } x_4 + x_6 \neq 1$$

$$v(2, 2, 4, 3) = 11 + \min(x_1 + x_2 + x_3 + 0 \cdot x_4 + 2x_5 - x_6)$$

$$\text{då } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$$

$$x_1, \dots, x_6 = 0/1$$

$$= 11 - 1 + 0 = 10 \text{ för } v(2, 2, 4, 3) = (0, 0, 0, 1, 0, 1)$$

Tinåten?

JA!

$$\Rightarrow z^* \in 6 + 8 \leq 14$$

$$\Rightarrow \left. \begin{array}{l} z^* \geq 9 \\ z^* \geq 10 \\ z^* \leq 14 \end{array} \right\} \Rightarrow \underline{\underline{10 \leq z^* \leq 14}}$$