Lektion 6

TANA21 – Beräkningsmatematik

Ordinära differentialekvationer

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Miniprojekt!

Begynnelsevärdesproblem

$$\begin{cases} y' = F(x, y) & (y = y(x),\ \text{oftast: } x = t = \text{tiden}) \\ y(0) = y_0 & x \geq 0 \end{cases}$$

- Diskretisera x-axeln: $x_i = 0 + ih$, $i = 0, 1, 2 \ldots$

Ex:

Euler framåt: Ersätt y' i (1) med $D_+ y(x_i)$:

$$\frac{y(x_i + h) - y(x_i)}{h} = F(x_i, y(x_i)) + O(h)$$

Sök $y_i \approx y(x_i)$ för $i = 1, 2, \ldots$

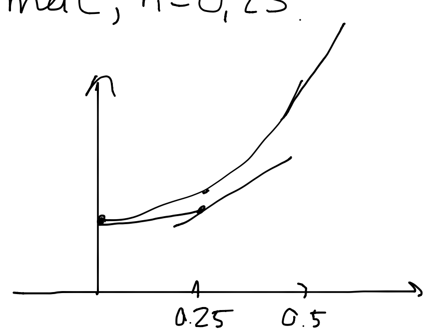
$$\frac{y_{i+1} - y_i}{h} = F(x_i, y_i)$$

Hela metoden:

- $y_0 = y(0)$ känd
- $y_{i+1} = y_i + h \cdot F(x_i, y_i)$ ← derivatan

Ex: Bestäm $y(0.5)$ med Euler framåt, $h = 0,25$

$$\begin{cases} y' = y^2 - x & (1) \\ y(0) = 1 \end{cases}$$



0, $y_0 = 1$, $x_0 = 0$

1) $y_1 = y_0 + 0,25\,(y_0^2 - x_0) = 1 + 0,25\,(1^2 - 0) = 1,25$

2) $y_2 = y_1 + 0,25\,(y_1^2 - x_1) = 1,25 + 0,25\,(1,25^2 - 0,25) = 1,578...$

$\therefore\ y(0,5) \approx y_2 \approx 1,58$

Heuns metod:

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

$k_1 = F(x_i, y_i)$, $k_2 = F(x_{i+1}, y_i + hk_i)$

9.1 (i) $= U_t = U_{xx}$ (ii) $- U_x = u_{xx}$ (iii) $y' = y^2$
(iv) $= y'' = 3y' + x^2$

a) Ordnar: iii, ii, iv

b) linjär: , iv, i, ii

c) första ordningen: , iii

9.2

a) $y'' = ye^{x^2} - 3y'$

$z = y'$

$y' = ye^{x^2} - 3z$

b) $a_3 y''' + a_2 y'' + a_1 y' + a y = g(x)$

$z = y'$
$z' = u$
$$u' = \frac{g(x) - a_2 u - a_1 z - a_0 y}{a_3}$$

c) $y'' = z,\ z'' = -y$

$u' = z \qquad y' = u$
$v' = -y \qquad z' = v$

9.5 Funktionen $f(x,y)$ är Lipschitzkont på $[a,b)$ om det finns en konstant L så att

$$|f(x,y) - f(x,\tilde{y})| \leq L\,|y - \tilde{y}|$$

för alla $x \in [a,b]$ och alla y och $\tilde{y}$ i $\mathbb{R}$

9.6 a) $f(x,y) = y + x + 1$ då $0 \leq x \leq 1$ och $-\infty < y < \infty$

$|(y + x + 1) - (\tilde{y} + x + 1)| < |y - \tilde{y}|$ så ja.

b) $-y^2$

$|(-y^2) - (\tilde{y}^2)| < |\tilde{y}^2 - y^2| = |(y + \tilde{y})(y - \tilde{y})|$

$= |y + \tilde{y}| \cdot |y - \tilde{y}|$ nej.

9.7 Begynnelseproblemet har entydig lösning om f är Lipschitzkont.

9,9 $y' = \cos x + \sin xy \qquad \frac{\pi}{2} < x < \pi$

$y\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} \qquad f(x,y) = \cos x + \sin xy$

$\frac{\partial f}{\partial y} = x \cos xy$

entydig existens och entydighet

Villkor är begränsad båda blir vilket följer Picards sats.

9,11 $y' = y + x \qquad 0 < x < 1$

$y(0) = 1$

a) Euler $h_0 = h = \frac{1}{2}$

funktionen

- $y_0 = 1, \; x_0 = 0$
- $y_1 = y_0 + 0{,}5(y_0^2 - x_0) = 1 + 0{,}5(1 + 0) = 1{,}5 \qquad x_1 = 0{,}5$
- $y_2 = y_1 + 0{,}5(y_1^2 - x_1) = 1{,}5 + 0{,}5(1{,}5 + 0{,}5) = 2{,}5 \qquad x = 1$

b) Heun

$y_{i+1} = y_i + \frac{h_i}{2}(k_1 + k_2) \Rightarrow$

$y_1 = y_0 + \frac{h_0}{2}(k_1 + k_2$

$k_1 = f(x_1, y_1) \qquad k_2 = f(x_1 + h_i, y_1 + h_i(f(x_1, y_1)))$

$\Rightarrow y_1 = 1 + \frac{1}{4}\left(1 + \left(\frac{1}{2} + \frac{3}{2}\right)\right) = 1 + \frac{1}{4} \cdot \frac{6}{2} = 1 + \frac{6}{8} = \frac{14}{8} = \frac{7}{4}$

$\frac{1}{2} + \frac{4}{2}$ $\quad x_1 = \frac{1}{2}$

$y \quad k_1$

$$y_2 = \frac{7}{4} + \frac{1}{4}\left(\frac{7}{4}+\frac{1}{2} + 1 + \frac{7}{4} + \frac{1}{2}\left(\frac{1}{2}+\frac{7}{4}\right)\right)$$

$$\frac{9}{4} + \frac{4}{4} + \frac{7}{4} + \frac{1}{4} + \frac{7}{8}$$

$$\frac{21}{4} + \frac{7}{8}$$

$$\frac{49}{8}$$

$$= \frac{7}{4} + \frac{49}{32} = \frac{56}{32} + \frac{49}{32} = \frac{105}{32}$$

e) $y_{i+1} = y_i + \frac{h_i}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = f(x_i, y_i)$

$k_2 = f\left(x_i + \frac{h_i}{2}, y_i + h_i \cdot \frac{k_1}{2}\right)$

$k_3 = f\left(x_i + \frac{h_i}{2}, y_i + h_i \frac{k_2}{2}\right)$

$k_4 = f(x_i + h_i, y_i + h_i k_3)$

$y_0 = 1, \; x_0 = 0, \; h = 0{,}5$

$y_1 = 1 + \frac{1}{12}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = \underline{\underline{1}}$

$k_2 = \frac{1}{2} + 1 + 1 \cdot \frac{1}{2} = \underline{\underline{2}}$

$$k_3 = \frac{1}{2} + 1 + 1 \cdot 1 = \underline{\frac{5}{2}}$$

$$k_4 = \frac{1}{2} + \frac{1}{2} + 1 \cdot \frac{5}{2} = \frac{9}{2}$$

$$y_1 = 1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1 + \frac{1}{12}\left(1 + 4 + \frac{10}{2} + \frac{9}{2}\right)$$

$$= \frac{24}{24} + \frac{1}{12}\left(\frac{2}{2} + \frac{8}{2} + \frac{10}{2} + \frac{9}{2}\right) = \frac{24}{24} + \frac{1}{6}\left(\frac{29}{2}\right) = \frac{12}{12} + \frac{29}{12}$$

$$= \frac{41}{12}$$

<u>9.13</u>

$$\begin{aligned} u' &= v \\ v' &= -u + x \end{aligned} \qquad \begin{aligned} 0 &< x < \frac{\pi}{2} \\ 0 &< x < \frac{\pi}{2} \end{aligned} \Rightarrow \begin{aligned} y' &= f(x,y) \\ y(0) &= c \end{aligned}$$

$$u(0) = 1 \qquad y = \begin{pmatrix} u \\ v \end{pmatrix};\; f\left(x, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} v \\ -u + x \end{pmatrix}$$

$$v(0) = 1 \qquad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y_{i+1} = y_i + h_i f(x_i, y_i)$$

a) $h_0 = h_1 = \frac{\pi}{4}$, $x_0 = 0$, $x_1 = \frac{\pi}{4}$ $x_2 = \frac{\pi}{2}$

$$y_0 = c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

$$y_1 = y_0 + h_0 f(x_0, y_0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + h_0 \begin{pmatrix} v_0 \\ -u_1 + x_0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\pi}{4} \\ 1 - \frac{\pi}{4} \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$y_2 = y_1 + h_1 f(x_1, y_1) = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + h_1 \begin{pmatrix} v_1 \\ -u_1 + x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{\pi}{4} + \frac{\pi}{4} + \frac{\cancel{\pi}^2}{16} \\ 1 - \frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi^2}{16} + \frac{\pi^2}{16} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\pi}{2} + \frac{\pi^2}{16} \\ 1 - \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

c) $y_{i+1} = y_i + \frac{h_i}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$u' = v$

$h_0 = \frac{\pi}{2}$, $x_0 = 0$, $v_0 = 1$, $u_0 = 1$

$v' = -u + x$

$$k_1 = f(x_0, y_0) = f\left(x_0, \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right) = \begin{pmatrix} v \\ -u \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$k_2 = f\left(x_i + \frac{h_i}{2}, y_i + h_i \cdot \frac{k_1}{2}\right) = f\left(\frac{\pi}{4}, 1 + \frac{\pi}{2}\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right)$$

$$= f\left(\frac{\pi}{4}, \begin{pmatrix} 1 + \pi/4 \\ 1 - \pi/4 \end{pmatrix}\right) = \begin{pmatrix} 1 - \pi/4 \\ -1 \end{pmatrix}$$

$$k_3 = f\left(x_i + \frac{h_i}{2}, y_i + \frac{h_i k_2}{2}\right) = f\left(\frac{\pi}{4}, 1 + \frac{\pi}{2}\begin{pmatrix} 1 - \pi/4 \\ -1 \end{pmatrix}\right)$$

$$= f\left(\frac{\pi}{4}, \begin{pmatrix} 1 - \pi/4 + \frac{\pi}{2} - \frac{\pi^2}{8} \\ -1 - \frac{\pi}{2} \end{pmatrix}\right) = f\left(\frac{\pi}{4}, \begin{matrix} 1 + \frac{\pi}{4} - \frac{\pi^2}{8} \\ -1 - \frac{\pi}{2} \end{matrix}\right)$$

$\underline{9.14}$ $y'' = y' - y + x \qquad 0 < x < 1$

$y'(0) = 0 \qquad u(0) = C$

$y(0) = -1$

$u = \begin{pmatrix} y \\ z \end{pmatrix}, \quad f\left(x, \begin{pmatrix} y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z - y + x \end{pmatrix} \qquad C = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$U_0 = C = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$

$z(0) = y'(0) = 0$

a) $y_0 = -1$

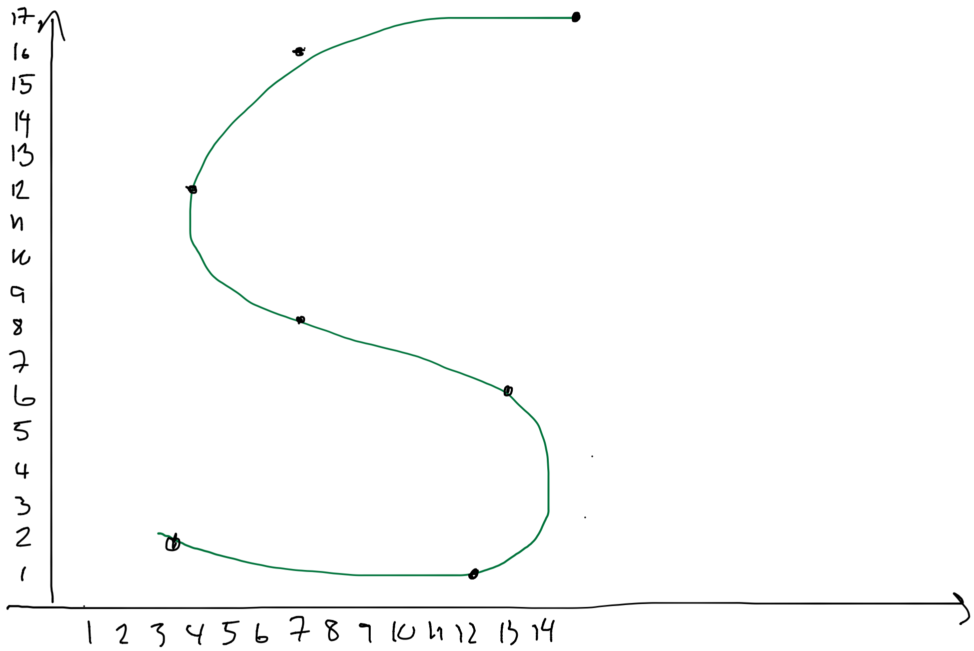
$u_1 = \begin{pmatrix} y_0 \\ u_0 \end{pmatrix} + h_0 f\left(x_0, \begin{pmatrix} y_0 \\ u_0 \end{pmatrix}\right)$

$= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{1}{2} f\left(0, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 + 0 \\ 0 + 0 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}$

$\Rightarrow u_2 = u_1 + h_1 f(x_1, u_1) = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \frac{1}{2} f\left(\frac{1}{2}, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}\right)$

$= \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1/2 \\ 1/2 + 1 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 + \frac{1}{4} \\ 1/2 + 1 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 3/2 \end{pmatrix} = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}$

$y_0 = -1, \; y_1 = -1 \quad y_2 = -3/4$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