Lektion 3

TAMS24 – Statistisk teori

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Lektionsgenomgång

ML: $L(\theta) = \begin{cases} \prod_{i=1}^{n} f(x_i;\theta) \\ \prod_{i=1}^{n} p(x_i;\theta) \end{cases}$ $\Rightarrow$ Find θ which maximize $L(\theta)$.

Note: 1° $\ln L(\theta)$: generally

Ex:

X with

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \quad \{x_1, \dots, x_n\}.$$

Find $\hat{\lambda}_{ML}$.

Solution

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ln L(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \boxed{\frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0} \qquad \hat{\lambda}_{ML} = \frac{1}{\bar{x}}$$

$$\frac{d^2 \ln L(x)}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$$

Ex:

$$X \sim U(-\theta, \theta) = Re(-\theta, \theta)$$

uniform distribution.

$\{x_1, \dots, x_n\}$

Find $\hat{\theta}_{ML}$.

Solution

$$f_X(x) = f(x;\theta) = \frac{1}{\theta-(-\theta)} = \frac{1}{2\theta}, \quad x \in [-\theta, \theta]$$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} = \left(\frac{1}{2\theta}\right)^n$$

$$\ln L(\theta) = \ln\left(\frac{1}{2}\right)^n - n\ln(\theta)$$

$$\cancel{\frac{d\ln L(\theta)}{d\theta} = \boxed{0 - \frac{n}{\theta} = 0}}$$

$$\frac{d\ln L(\theta)}{d\theta} = -\frac{n}{\theta} < 0$$

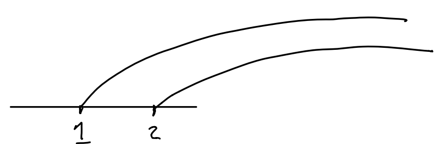
$$\ln L(\theta) \downarrow \quad L(\theta) \downarrow \quad \theta_{min} \Rightarrow L_{max}(\theta)$$

$\theta_{min} = ?$

$$\begin{cases} -\theta \le x_1 \le \theta \\ \quad \vdots \\ -\theta \le x_n \le \theta \end{cases} \Leftrightarrow \begin{cases} |x_1| \le \theta \\ |x_2| \le \theta \\ \quad \vdots \\ |x_n| \le \theta \end{cases}$$

$$\begin{cases} 1 \le \theta \\ 2 \le \theta \end{cases} \Leftrightarrow 2 \le \theta$$

$$\theta_{min} = 2 = \max\{2,1\}$$



$$\theta_{min} = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

- Standard error (medelfelet) = estimate of $D(\hat{\Theta})$

- $X \sim Po(\mu)$,

$$p_X(k) = \frac{\mu^k}{k!} e^{-\mu}, \quad k = 0, 1, 2, 3 \ldots$$

11.23

$x = 16$

$X \sim Bin(25, p)$

a) Skattning av p

$$p_X(k) = p_X(k, p) = \binom{25}{k} p^k (1-p)^{25-k}, \quad k = 0, \ldots, 25$$

$$L(p) = p(x; p) = \binom{25}{x} p^x (1-p)^{25-x}$$

$$\ln(L(p)) = \ln\binom{25}{x} + x \ln p + (25-x)\ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = 0 + \frac{x}{p} - \frac{25-x}{1-p} = 0 \iff$$

$$\frac{x(1-p) - p(25-x)}{p(1-p)} = 0 \iff x(1-p) - p(25-x) = 0$$

$$p = \frac{x}{25}$$

$$\frac{d^2(\ln L(p))}{dp^2} = -\frac{x}{p^2} - \frac{25-x}{(1-p)^2} < 0$$

$$\Rightarrow \hat{p}_{ML} = \frac{x}{25} = \frac{16}{25} = 0.64$$

b) $\hat{p} = \frac{X}{25}$

$V(\hat{p}) = V\left(\frac{X}{25}\right) = V(X) \cdot \left(\frac{1}{25}\right)^2 = \frac{1}{25^2} \cdot 25\, p(1-p) =$

$= \frac{p(1-p)}{25}$

$D(\hat{p}) = \sqrt{\frac{p(1-p)}{25}} \left(\approx \sqrt{\frac{\hat{p}(1-\hat{p})}{25}}\right)$

c) $D(\hat{p}) = \sqrt{\frac{p(1-p)}{25}} \approx 0.096$

1.10

Diskret s.v X.

$p_X(k) = \theta(1-\theta)^{k-1}, \quad k = 1,2,3,\dots, \quad 0<\theta<1$

Slumpmässigt stickprov: $\{4,5,4,6,4,1\}$

$L(\theta) = \prod_{i=1}^{6} \theta(1-\theta)^{k-1} = \theta(1-\theta)^{4-1} \cdot \theta(1-\theta)^{5-1} \cdot$

$\cdot\, \theta(1-\theta)^{4-1} \cdot \theta(1-\theta)^{6-1} \cdot \theta(1-\theta)^{4-1} \cdot \theta(1-\theta)^{1-1} =$

$= \theta^6 (1-\theta)^{18}$

b) $\ln L(\theta) = 6\ln\theta + 18\ln(1-\theta)$

$\frac{d \ln L(\theta)}{d\theta} = \frac{6}{\theta} - \frac{18}{1-\theta} = 0$

$\Rightarrow \quad 6(1-\theta) - 18\theta = 0 \quad \Leftrightarrow \quad 6 - 6\theta - 18\theta = 0$

$\Leftrightarrow \quad \theta = \frac{6}{24} = \frac{1}{4}$

11.12

$X \sim Po(\mu)$

n dagar.

$\{x_1, ..., x_n\}$ anrop

a) ML-skattningen av μ.

$$p_X(k) = \frac{\mu^k}{k!} e^{-\mu}, \quad k = 0,1,2,...$$

$$p_X(k,\mu) = \frac{\mu^k}{k!} e^{-\mu}$$

$$L(\mu) = p(x;\mu) = \prod_{i=1}^{n} \frac{\mu^{x_i}}{x_i!} e^{-\mu} = \frac{\mu^{\sum_{i=1}^{n} x_i}}{\sum_{i=1}^{n} x_i!} e^{-\mu}$$

$$\ln(L(\mu)) = \sum x_i \ln \frac{\mu}{\sum x!} - \mu \ln e = \sum x (\ln \mu - \ln \sum x!) - n\mu$$

$$\frac{d \ln(L(\mu))}{d\mu} = \frac{\sum_{i=1}^{n} x}{\mu} - n = 0 \iff \underline{\bar{x} = \hat{\mu}}$$

$$\frac{d^2 \ln(L(\mu))}{d\mu} = - \frac{x}{\mu^2} < 0$$

b) $$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu = \frac{1}{n} \cdot n\mu$$

$$V(\bar{X}) = V\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) = \{\text{oberoende}\} =$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \underbrace{V(X_i)}_{\mu} = \frac{\mu}{n}$$

$$\Rightarrow D(\bar{X}) = \sqrt{\frac{\mu}{n}}$$

c) $\hat{y}_{obs} = 100.875$ $\left(\frac{115+82+\ldots+92}{8} \right)$

d $D = \sqrt{\frac{\mu}{n}} = \underline{\underline{3.55}}$

11.14

$\{x_1, x_2, \ldots, x_n\}$

$f_X(x) = \theta x^{\theta-1}, \quad 0 < x < 1.$

Ange ML-uppskattningen av θ

$$L(\theta) = \prod_{i=1}^{n} \theta x^{\theta-1} = \theta^n \prod_{i=1}^{n} x^{\theta-1}$$

$$\ln L(\theta) = n \ln\theta + (\theta-1) \ln \prod_{i=1}^{n} x$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} + \ln \prod_{i=1}^{n} x_i = 0$$

$$\Leftrightarrow \theta = -\frac{n}{\ln \prod_{i}^{n} x_i}$$

$$\underline{\underline{\frac{d^2 \ln L(\theta)}{d\theta^2} < 0}}$$