Föreläsning 7

TAMS24 – Statistisk teori

Hypotesptövning vid normalfördelning
Samband mellan test och konfidensintervall

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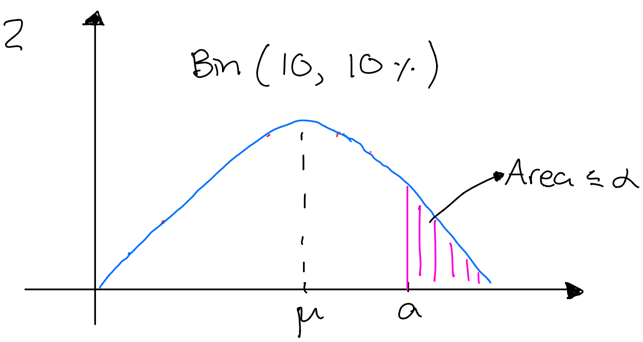
Ex:

Students think that the smoke rate of students at LiU is bigger than 10%. Ask 10 students there are smokers. Test the hypothesis that students are right with significance level 5%.

$$H_0: \ p = 10\% \quad \text{vs.} \quad H_1: \ p > 16\%$$

$$X \sim Bin(10, p), \quad x = 2$$



a) C-method

$$P(Bin(10, 10\%) \geq a) \leq \alpha$$

$a = 4 \Rightarrow$

$$P(Bin(10, 10\%)) \geq 4) = 1.28\% < 5\%$$

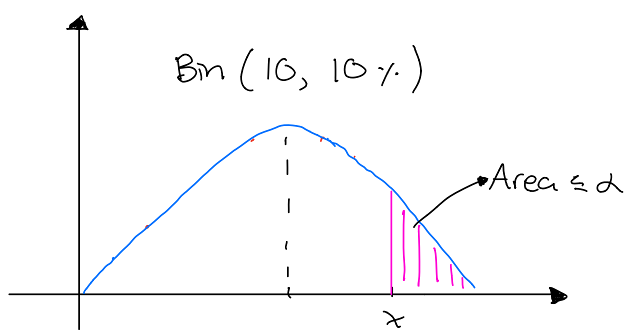
Rejection / critical region

$$C = [a, \infty) = [4, \infty].$$

$$2 \notin C,$$

Don't reject H_0!

b) p-value



$$\text{p-value} = P(Bin(10, 10\%) \geq x) = 26.39\% > \alpha$$

Don't reject H_0!

p-value $< \alpha \iff$ reject H_0!

$X \in C \iff$ reject H_0!

c) Find the power if true

$p = 15\%$

$h(15\%) = P(\text{reject } H_0 \text{ if } H_0 \text{ is false}, p = 15\%) =$

$= P(\underline{X} \in C, \text{ if } \overset{\text{true value}}{p} = 15\%) = P(Bin(10, 15\%) \geq 4) =$

$= \{\text{table}\} = 4.99\%$

HT - Type 2

With Normal distribution (approximation)

Test statistic, TS

$\begin{cases} \text{fact from CI} \\ H_0 \end{cases}$

Region/critical region, C

$\begin{cases} \text{fact from CI} \\ H_1 \end{cases}$

If $TS \in C$, reject H_0

1. $X \sim N(\mu, \sigma)$, σ is known

fact:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

(1.1) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu > \mu_0 \end{cases}$

$(1-\alpha)$ lower limit CI for μ:

$$I\mu = (a, \infty) = \left(\bar{x} - \lambda_\alpha \cdot \frac{\sigma}{\sqrt{n}}, \infty\right)$$

i.e

$$y > \bar{x} - \lambda_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

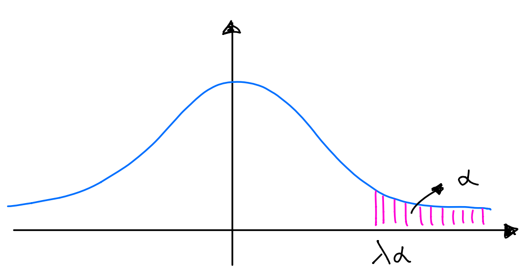
"reject H_0" = "H_1 is true" = $\boxed{\bar{x} - \lambda_\alpha \cdot \frac{\sigma}{\sqrt{n}} > \mu_0}$

$$\bar{x}_0 - \mu_0 > \lambda_\alpha \cdot \frac{\sigma}{\sqrt{n}} \Rightarrow \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > \lambda_\alpha$$

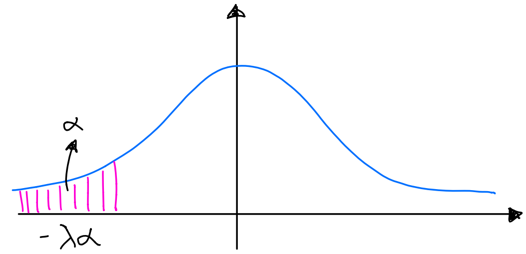
$$TS = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}},$$

$$C = (\lambda_\alpha, \infty)$$

$TS \in C \Rightarrow$ reject H_0



(1.2) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu < \mu_0 \end{cases}$

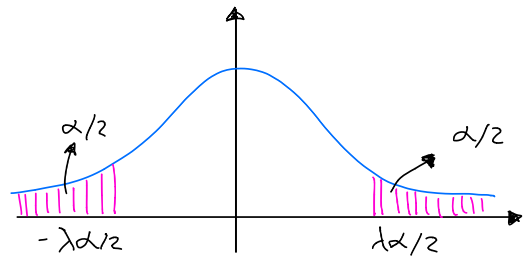


$TS = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$,

$C = (-\infty, -\lambda\alpha)$

$TS \in C \Rightarrow$ reject H_0

(1.3) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \neq \mu_0 \end{cases}$



$TS = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$,

$C = (-\infty, -\lambda\alpha/2) \cup (\lambda\alpha/2, \infty)$

$TS \in C \Rightarrow$ reject H_0 !

Ex 7.1:

$X \sim N(\mu, 0.2)$, $\{x_1, \dots, x_{11}\}$: $\bar{x} = 4.2$

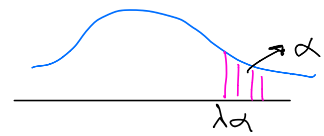
$H_0: \mu = 4$ vs H_1 $\mu > 4$, $\alpha = 5\%$

a) Do we reject H_0?

fact:

$\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$, $\sigma = 0.2$

$$TS = \frac{\bar{x} - 4}{\sigma/\sqrt{n}} \approx 3.32$$



$$C = (\lambda_{0.05}, \infty) = \{table\} = (1.645, \infty).$$

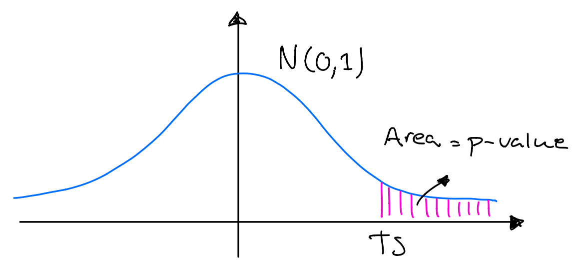
$TS \in C$, reject H_0.

i.e.

$y > 4$.

Significant!

b) p-value method



$$\text{p-value} = P(N(0,1) \geq TS) =$$

$$= P(N(0,1) \geq 3.32) = 1 - \Phi(3.32) \approx 0.05\% < \alpha$$

p-value $< \alpha \Rightarrow$ reject H_0.

c) Find the power if true $\mu = 5$.

$$h(5) = P(\text{reject } H_0 \text{ if } H_0 \text{ is false}, \mu = 5) =$$

$$= P(TS \in C \text{ if } \mu = 5) = P\left(\frac{\bar{X} - 4}{\sigma/\sqrt{n}} > \lambda_{0.05}, \text{ if } \mu = 5\right) =$$

$=$

Note:

$$\frac{\bar{X} - 5}{\sigma/\sqrt{n}} \sim N(0,1), \text{ Not } \frac{\bar{X} - 4}{\sigma/\sqrt{n}}$$

$=$

$$= P\left(\frac{\bar{X} - 5}{\sigma / \sqrt{n}} > \frac{4 + \lambda_{0.05} \cdot \frac{\sigma}{\sqrt{n}} - 5}{\sigma / \sqrt{n}} \right) = P(N(0,1) > -14.94) =$$

$$= \Phi(14.94) \simeq 1$$

Ex 7.2:

$$\left.\begin{array}{l} X \sim N(\mu_1, 0.3) \\ Y \sim N(\mu_2, 0.4) \end{array}\right\} \text{independent}$$

$n_1 = 10, \quad \bar{x} = 3.5$

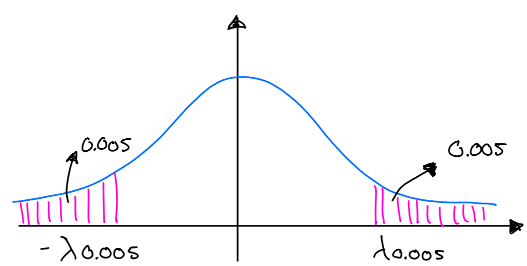
$n_2 = 10, \quad \bar{x} = 3.2$

$H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$; $\alpha = 1\%$

a) C-method

fact:

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$



$$TS = \frac{(\bar{x} - \bar{y}) - (0)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \approx 1.90$$

$$C = (-\infty, -\lambda_{0.005}) \cup (\lambda_{0.005}, \infty) =$$

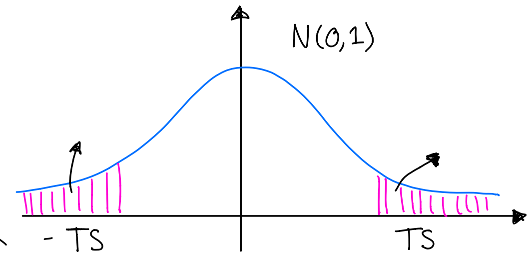
$$= (-\infty, -2.58) \cup (2.58, \infty)$$

$TS \notin C$, don't reject H_0. Not significant.

b) p-value

p-value $= 2P(N(0,1) \geq TS) =$

$= 2P(N(0,1) \geq 1.9) \approx 5.74\% > \alpha$



Don't reject H_0.

c) Calculate Type II error if true

$\mu_1 - \mu_2 = 0.6$

$\beta(0.6) = P($ don't reject H_0 if H_0 is false,

if $\mu_1 - \mu_2 = 0.6) = P(TS \notin C$, if $\mu_1 - \mu_2 = 0.6) =$

$$= P\left(\frac{(\bar{X} - \bar{Y}) - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq 2.58, \text{ if } \mu_1 - \mu_2 = 0.6\right)$$

$=$ Note: $=$

$$\frac{(\bar{X} - \bar{Y}) - 0.6}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

$$= P\left(-2.58 < \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < 2.58, \text{ if } \mu_1 - \mu_2 = 0.6\right)$$

$= P(-6.37 < N(0,1) < -1.22) = 0.11 = 11\%$

$\sigma^2 \longrightarrow \chi^2$

Ex 7.3:

$X \sim Po(\mu)$, $x_1 = 10$, $x_2 = 20$

$$\begin{cases} H_0: \mu = 14 \\ H_1: \mu > 14 \end{cases}, \quad \alpha = 5\% .$$

Do you reject H_0?

Solution:

$$n\hat{\mu} = n\bar{x} = 2 \cdot \frac{10+20}{2} = 30 > 15, \quad \hat{\mu} = \bar{x}$$

fact:

$$\frac{\bar{X} - \mu}{\sqrt{\mu / n}} \approx N(0,1)$$

$$TS = \frac{\bar{x} - 14}{\sqrt{\bar{x}/n}} \approx 0.38, \quad C = (\lambda_{0.05}, \infty) = (1.645, \infty)$$

$TS \notin C$, don't reject H_0!