Föreläsning 4

TAMS24 – Statistisk teori

χ^2 - fördelning och t-fördelning
Konfidensintervall
Prediktionsintervall

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$X \quad \theta \quad \hat{\theta}$

Interval estimation:

$\theta \in (a, b)$

e.g.

$X = \{\text{lifetime of lions}\}$

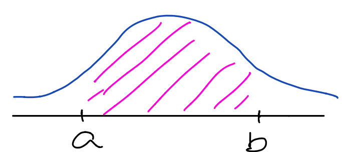
$X \sim Exp\left(\frac{1}{\mu}\right)$, $E(X) = \mu$, expected lifetime

$I_\mu = (15\text{years}, 25\text{ years})$

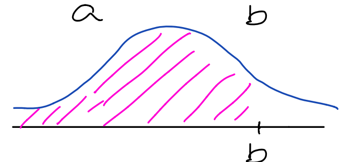
1. Preparation

(1) X is continous

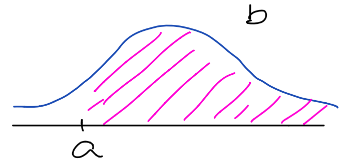
$P(a \le X \le b) = $ Area of



$P(X \le b) = $ (Lef) area of



$P(X \le b) = $ (right) area of



(2) $X_1, X_2, \ldots, X_n$ i.i.d (independent and identicaly distributed) and each

$X_i \sim N(\mu, \sigma)$, $i = 1, 2, \ldots, n$

Then

$$\bar{X} = X_1 + \dots X_n \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Since

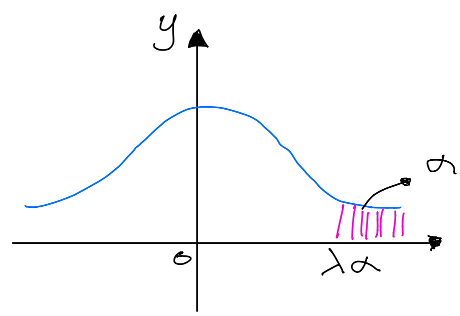
$$E(\bar{X}) = \mu, \quad V(\bar{X}) = \frac{\sigma^2}{\sqrt{n}}$$

Note:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

(3) Notations:

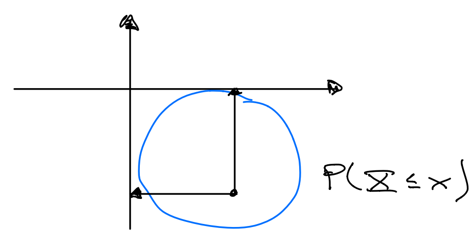
(3.1) $X \sim N(0,1)$



$$P(X \geq \lambda_\alpha) = \alpha$$

e.g.

$$\lambda_{0.05} = \frac{1.64 + 1.65}{2} = 1.645$$



left $= 1 - 0.05 = 0.95$

$$0.95 = \frac{0.9495 + 0.9505}{2}$$

(3.2) $X \sim t(f)$:

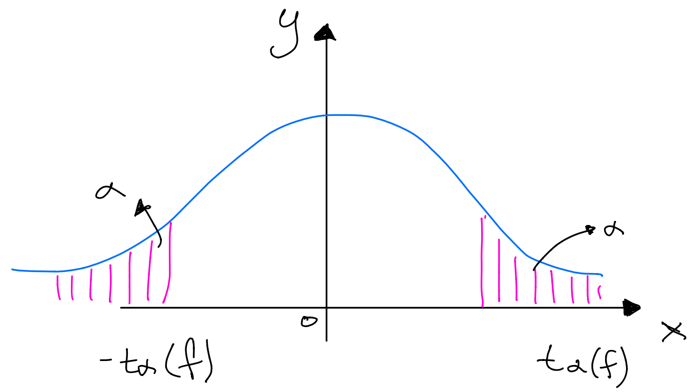
t-distribution.

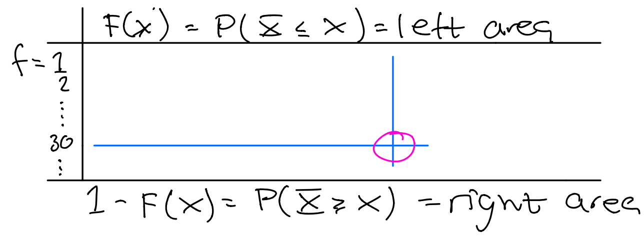
f = degrees of freedom (frihetsgrader)

$P(X \geq t_\alpha(f)) = \alpha$

$\alpha = 0.05, \ f = 30$

$t_{0.05}(30) = 1.7$





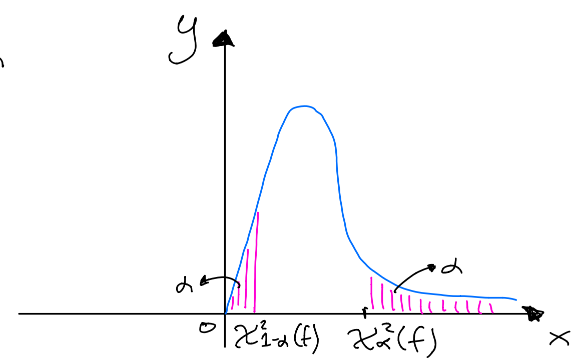
(3.3) $X \sim \chi^2(f)$: Chi-square distribution

f = degrees of freedom

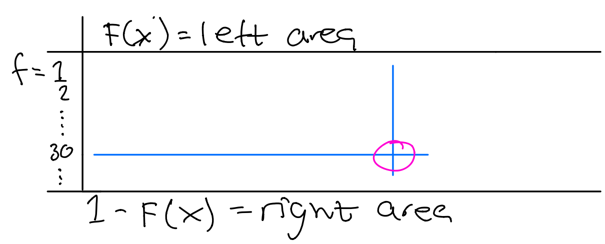
$P(X \geq \chi^2_\alpha(f)) = \alpha$

$\alpha = 0.05, \ f = 30$

$\chi^2_{0.05}(30) \overset{\text{table}}{=} 43.78$



$\chi^2_{1-\alpha}(f) = \chi^2_{0.95}(30) = 18.49$



2. Interval Estimation

Type I:

One sample

$$\{X_1, X_2, \dots, X_n\} \sim N(\mu, \sigma)$$

(1) $(1-\alpha)$

$I_\mu = (0, 300\,cm) \Rightarrow 100\%$

$I_\mu = (175, 185) \Rightarrow ?$

confidence intevall for μ: I_μ.

$1-\alpha$ = confidence coefficient (konfidensgrad)

e.g.

$1-\alpha = 95\%$,

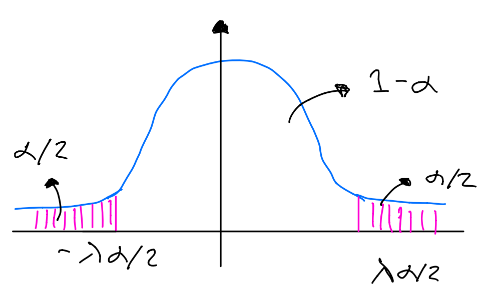
95% confidence to say

$\mu \in I_\mu$:

$$100 \text{ samples}: \begin{cases} 95: & \mu \in I_\mu \\ 5: & \mu \notin I_\mu \end{cases}$$

(1.1) If σ is known, fact:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$



$$P\left(-\frac{\lambda}{2} \le \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le -\frac{\lambda}{2}\right) = 1-\alpha$$

$$P\left(-\frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}} \le \bar{X} - \mu \le \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

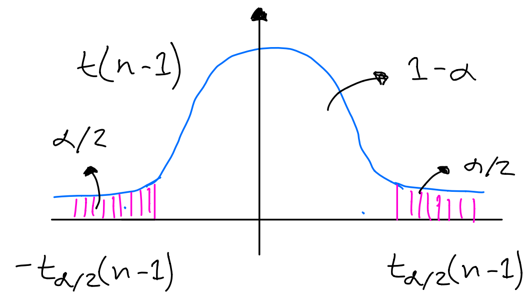
$$P\left(\bar{x} - \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$(1-\alpha)\ CI:\ I_\mu = \left(\bar{x} - \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}}\right) =$$

$$= \bar{x} \mp \frac{\lambda\alpha}{2} \cdot \frac{\sigma}{\sqrt{n}}$$

(1.2) σ is known, fact:

$$\frac{\bar{x}-\mu}{S/\sqrt{n}} \sim t(n-1)$$



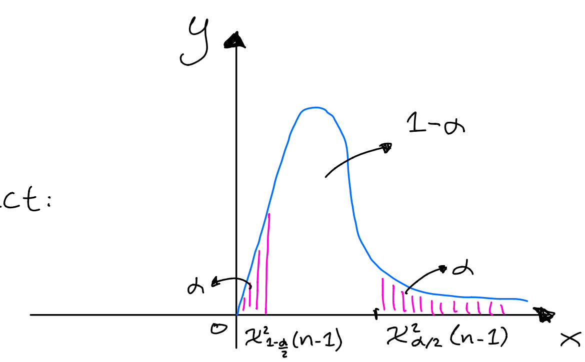
$$P\left(-t_{\alpha/2}(n-1) \leq \frac{\bar{x}-\mu}{S/\sqrt{n}} \leq t_{\alpha/2}(n-1)\right) = 1-\alpha$$

$$(1-\alpha)\ CI:\ I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{S}{\sqrt{n}}$$

$$I_\mu = (a, \infty) = \left(\bar{x} - \lambda\alpha \cdot \frac{\sigma}{\sqrt{n}},\ \infty\right)$$

Note:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$



(1.3) $(1-\alpha)$ CI for σ^2 or σ; fact:

$$\frac{(n-1)S^2}{\sigma} \sim \chi^2(n-1)$$

$$P\left(\chi^2_{1-\frac{\alpha}{2}}(n-1) \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha/2}(n-1)\right) = 1-\alpha$$

$(1-\alpha)$ CI:

$$I\sigma^2 = \left(\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}(n-1)},\ \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}\right)$$

$(1-\alpha)$ CI:

$$I_\sigma = \left(\sqrt{\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}} \right)$$

Ex 4.1:

The heights of a certain kind of christmas trees are assumed to be $N(\mu, \sigma)$

Sample:

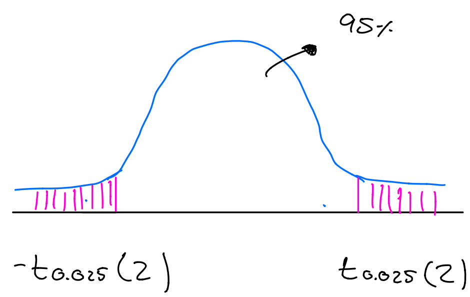
178cm, 182cm, 186 cm.

Find confidence intervall for expected height of a tree. — I_μ.

Solution:

Fact:

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$



$$I_\mu = \bar{x} \mp t_{0.025}(2) \cdot \frac{S}{\sqrt{n}} = \dots \approx (172, 192)$$

$\bar{x} = 182$ cm, $t_{0.025}(2) = 4.3$, $S^2 = 16$, $s = 4$

Remark:

① two sided (tvåsidigt) CI: (a, b)

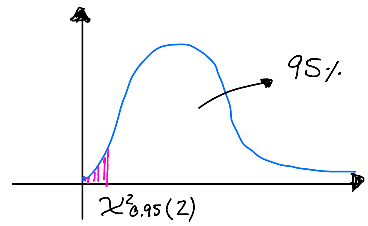
② one sided (ensidigt) CI: $\begin{cases} (-\infty, b), \text{ upper limit CI} & \text{uppåt begränsat} \\ (a, \infty), \text{ lower limit CI} & \text{nedre begränsat} \end{cases}$

Ex 4.1':

Find 95% upper limit CI for σ^2?

Solution:

Fact:



$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

$$I_{\sigma^2} = (-\infty, b) = \left(0, \frac{(n-1)S^2}{\chi^2_{1-\alpha}(n-1)}\right)$$

$$P\left(\frac{(n-1)S^2}{\sigma^2} \geq \chi^2_{0.95}(2)\right) = 95\%$$

$$I_{\sigma^2} = \left(0, \frac{(n-1)S^2}{\chi^2_{0.95}(2)}\right) \approx (0, 320), \quad \chi^2_{0.95}(2) = 0.1$$

Remark:

A short cut to get one-sided from two-sided CI.

I. Get the corresponding form

II Replace $\frac{\alpha}{2}$ by α.

Note: If it is upper limit for σ^2 for $\chi^2(f)$. Use $(0, b)$.

3. Theorems

(3.1) Fact:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Theorem 1:

$$X_1, \dots, X_n$$

i.i.d & each

$$X_i \sim N(0,1),$$

then

$$\sum_{i=1}^{f} X_i^2 = \chi^2(f)$$

Theorem 2:

If

$$X \sim \chi^2(f_1)$$

and

$$Y \sim \chi^2(f_2)$$

are independent, then

$$X + Y \sim \chi^2(f_1 + f_2)$$

Proof:

$$\frac{(n-1)S^2}{\sigma^2} = \frac{(n-1) \cdot \frac{1}{(n-1)} \cdot \sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2} =$$

$$= \frac{\sum_{i=1}^{n} (X_i - \mu + \mu - \bar{X})^2}{\sigma^2} = \dots = \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma^2} - \frac{n(\bar{X} - \mu)^2}{\sigma^2} =$$

$$= \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma^2}\right)^2}_{\chi^2(n)} - \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2(1)} = \chi^2(n-1)$$