Lösningsgång

TAOP07 – Optimeringslära grundkurs

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Råvara	Produkt P_1	P_2
R_1	2	1
R_2	1	1
R_3	1	0

Scenario	Pris per ton P_1	P_2
1	15	10
2	10	15
3	12	12

Råvaror: R_1, R_2, R_3

Produkter: P_1, P_2

$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\}$ antal tillverkade av P_1, P_2

$$\max \min \{15x_1 + 10x_2,\ 10x_1 + 15x_2,\ 12x_1 + 12x_2\}$$

$$2x_1 + x_2 \le 1500$$

$$x_1 + x_2 \le 1200$$

$$x_1 \le 500$$

$$x_1, x_2 \ge 0$$

$\Rightarrow$ $\max z$

då
$$\begin{cases} z \le 15x_1 + 10x_2 \\ z \le 10x_1 + 15x_2 \\ z \le 12x_1 + 12x_2 \\ 2x_1 + x_2 \le 1500 \\ x_1 + x_2 \le 1200 \\ x_1 \le 500 \end{cases}$$

$$x_1, x_2 \ge 0$$

(2) $\max z = 4x_1 + 3x_2$

Baslösningens variabler ≥ 0

c, g tas bort.

Analogt $\Rightarrow$ iv och vi tas bort då de innehåller c, g.

i, iii, kan bara byta en variabel per iteration

$\Rightarrow$ ii, v kan uppträda

(3) $\max z = 18x_1 + 6x_2 + 15x_3 + 24x_4$

då

$$\text{(P)} \quad \begin{array}{l|l} 4x_1 + x_2 + x_3 + 8x_4 \geq 7 & y_1 \\ x_1 + x_2 + 3x_3 + 2x_4 \geq 4 & y_2 \end{array}$$

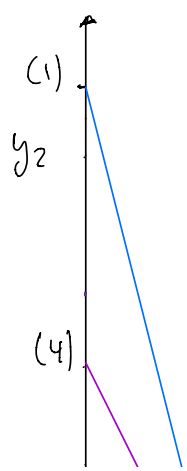
$x_1, \dots, x_4 \geq 0$

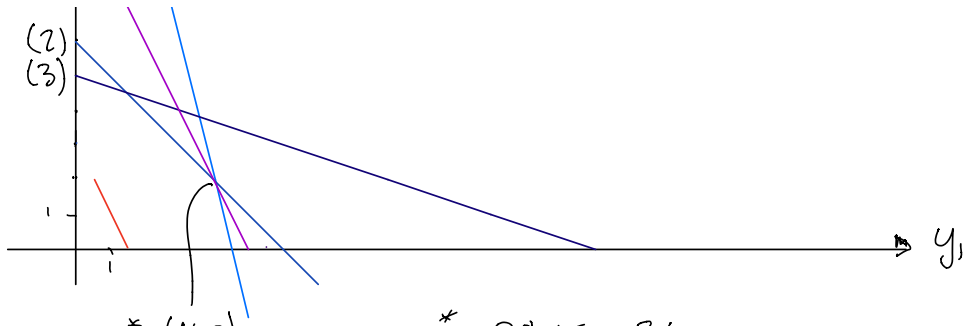
$\Leftrightarrow \min w = 7y_1 + 4y_2$

$$\text{(D)} \quad \begin{array}{ll} 4y_1 + y_2 \leq 18 & (1) \\ y_1 + y_2 \leq 6 & (2) \\ y_1 + 3y_2 \leq 15 & (3) \\ 5y_1 + 2y_2 \leq 24 & (4) \\ y_1, y_2 \geq 0 & \end{array}$$

$y_2 = 12 - \frac{5y_1}{2}$

$y_1 = \frac{24}{5}$





$y^* = (4,2) \Rightarrow \qquad w^* = 28 + 8 = 36$

b) $x_1^*(Ay^* - b) =$

$x_1^*(4y_1 + y_2 - 18) = 0 \Rightarrow$ inget ty $(\ldots) = 0$

$x_2^*(y_1 + y_2 - 6) = 0 \Rightarrow$ inget

$x_3^*(y_1 + 3y_2 - 15) = 0 \Rightarrow x_3^* = 0$

$x_4^*(5y_1 + 2y_2 - 24) = 0 \Rightarrow$ inget

$y_1^*(b - Ax^*) =$

$y_1^*(7 - 4x_1 - x_2 - x_3 - 5x_4) = 0$

$y_2^*(4 - x_1 - x_2 - 3x_3 - 2x_4) = 0$

$$\Rightarrow \begin{cases} 4x_1 + x_2 + 5x_4 = 7 \\ x_1 + x_2 + 2x_4 = 4 \end{cases} \sim \left(\begin{array}{ccc|c} 4 & 1 & 5 & 7 \\ 1 & 1 & 2 & 4 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 0 & -3 & -3 & -9 \\ 1 & 1 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -3 & -3 & -9 \\ 3 & 0 & 3 & -5 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Tillåten?

$$\begin{cases} 1-t \geq 0 & \Rightarrow \quad t \leq 1 \\ 3-t \geq 0 & \quad t \leq 3 \\ t \geq 0 & \quad t \geq 0 \end{cases} \quad \Rightarrow \quad 0 \leq t \leq 1$$

(4) $z^* = \max \; 4x_1 + 3x_2$

då $12x_1 + 4x_2 \leq 45$

$4x_1 + 8x_2 \leq 35$

$x_1, x_2 \geq 0$, och heltal

$\Rightarrow x_1 = \frac{35}{4} - 2x_2$

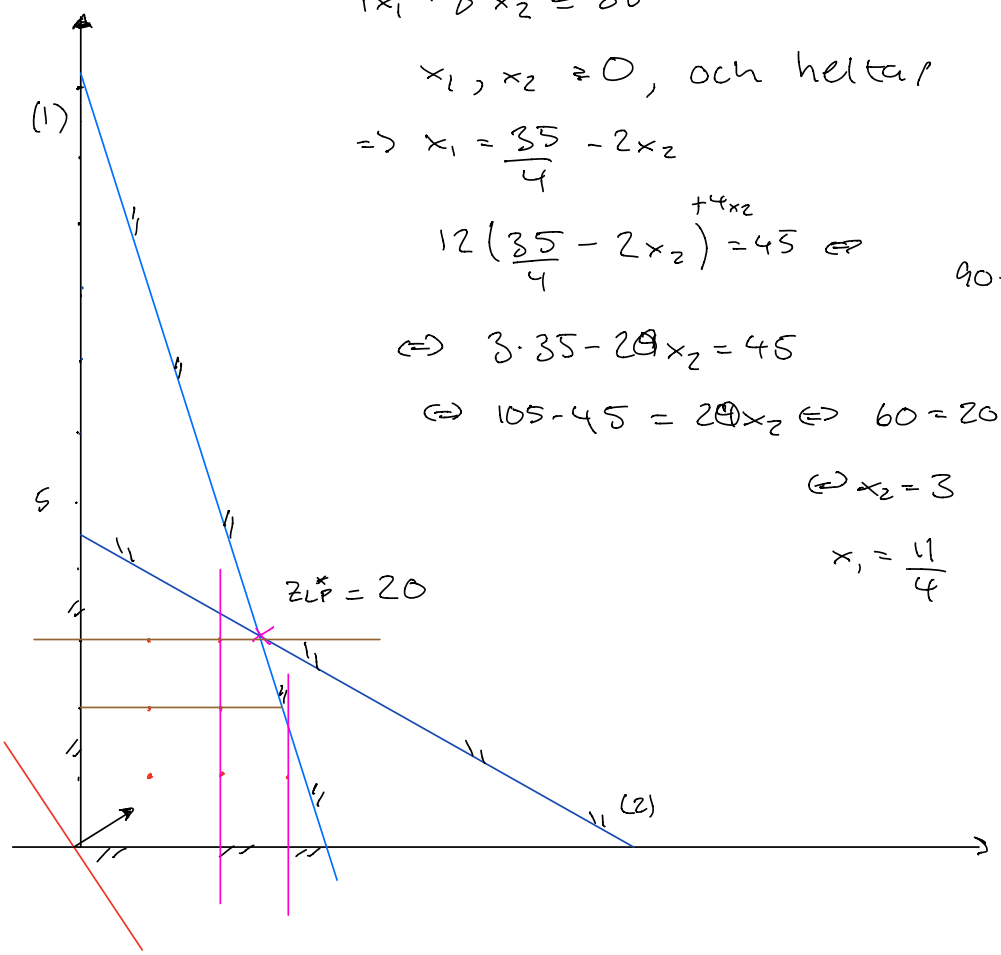
$12\left(\frac{35}{4} - 2x_2\right) + 4x_2 = 45 \Leftrightarrow$ $\quad 90 + 15 = 105$

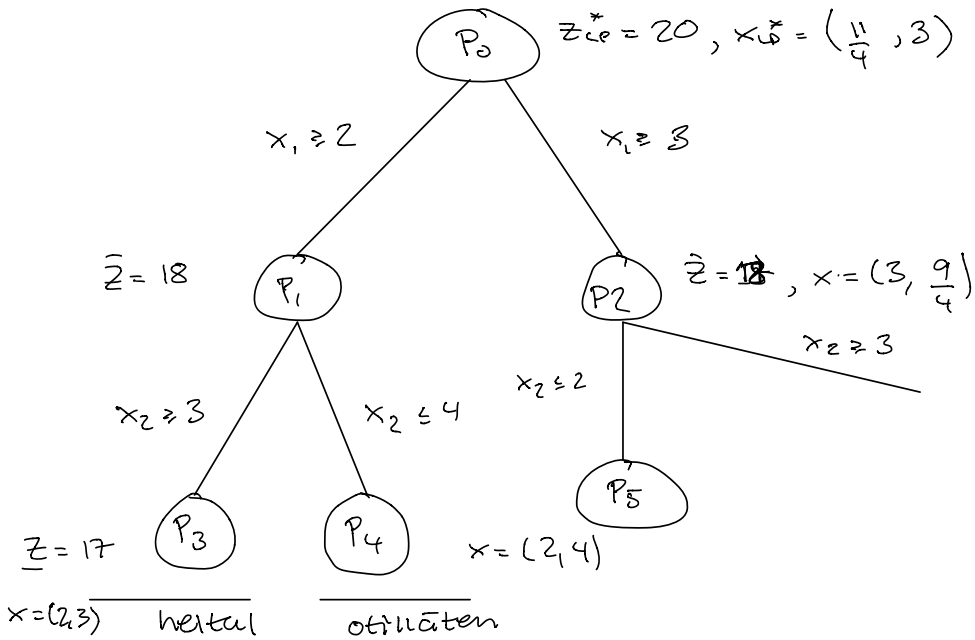
$\Leftrightarrow 3 \cdot 35 - 20x_2 = 45$

$\Leftrightarrow 105 - 45 = 20x_2 \Leftrightarrow 60 = 20x_2$

$\Leftrightarrow x_2 = 3$

$x_1 = \frac{11}{4}$





P_1 P_0 och $x_1 = \lfloor \frac{11}{4} \rfloor \geq 2$

P_2 P_0 och $x_1 = \lfloor \frac{11}{4} \rfloor + 1 \leq 3$

Undersöker villkor 2, $x_1 \geq 2$

$$\left.\begin{array}{l} x_1 = 2 \Rightarrow 35 - 4 \cdot 2 = 27 \\ x_2 = \frac{27}{8} \Rightarrow 27 - \frac{27}{8} \cdot 8 \end{array}\right\} \Rightarrow \bar{z} = 18$$

P_3 P_1 och $x_2 = \lfloor \frac{27}{8} \rfloor \geq 3$

P_4 P_1 och $x_2 = \lfloor \frac{27}{8} \rfloor + 1 \leq 4$

Undersöker villkor 1

$$\left.\begin{array}{l} x_1 = 3 \Rightarrow 45 - 36 = 9 \\ x_2 = \frac{9}{4} \Rightarrow 9 - \frac{9}{4} \cdot 4 = 0 \end{array}\right\} \Rightarrow z^*_{LP} = 12 + \frac{27}{4} = \frac{48 + 27}{4}$$

$$\Rightarrow \bar{z} = 18$$

P_5 : P_2 och $x_2 = \lfloor \frac{9}{4} \rfloor = 2$ $\Rightarrow$

P_6 : P_2 och $x_2 = \lfloor \frac{9}{4} \rfloor + 1 = 3$

Undersöker vinkel 1

$x_2 = 2 \Rightarrow 45 - 8 = 37$

$x_1 = \frac{37}{12} \Rightarrow 37 - \frac{37}{12} = 0$

5) $f(x) = (x_1 - 2)^4 + (x_1 - 2x_2)^2$

a) $\nabla f(x) = \left(4(x_1-2)^3 + 2(x_1 - 2x_2),\ -4(x_1 - 2x_2)\right)$

$\bar{x} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow \nabla f(\bar{x}) = \begin{pmatrix} 4 \cdot (-8) - 2 \cdot 6 \\ -4 \cdot (-6) \end{pmatrix} = \begin{pmatrix} -44 \\ 24 \end{pmatrix}$

$\Rightarrow \nabla f(\bar{x})^T \bar{d} = (-44, 24) \begin{pmatrix} 2 \\ -1 \end{pmatrix} < 0 \Rightarrow$ avtaganderiktning

b) $x(t) = \bar{x} + t\bar{d} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2t \\ 3-t \end{pmatrix}$, $t > 0$

$\varphi(t) = (2t - 2)^4 + (\underbrace{2t - 2(3-t)}_{2t - 6 + 2t = 4t - 6})^2$

$\varphi'(t) = 8(2t-2)^3 + 8(4t-6)$

$\varphi''(t) = 192(2t-2)^2 + 32 > 0 \Rightarrow$ konvex

$\varphi'(t) = 64(t-1)^3 + 8(4t-6) = 0$

$\Leftrightarrow 8(t^2 - 2t + 1)(t-1) + 4t - 6 = 0$

$\Leftrightarrow 8(t^3-2t^2+t-t^2+2t-1)+4t-6$

$\Leftrightarrow 8t^3-24t^2+24t-8+4t-6=0 \Leftrightarrow 8t^3-24t^2+28t-14=0$

$$\Rightarrow \left.\begin{array}{l} \varphi'(1) = -162 < 0 \\ \varphi(t) \text{ konvex} \end{array}\right\} \Rightarrow \text{Optimala steget} \geq 1$$

c) $-\nabla^2 f(\bar{x})^{-1} \nabla f(\bar{x})$

$$f(\bar{x}) = \begin{pmatrix} -44 \\ 24 \end{pmatrix}$$

$f''_{x_1x_1} = 12(x_1-2)^2+2$ $\qquad f''_{x_1x_2} = -4$

$f''_{x_2x_2} = -8$

$$f''_{x_1x_1}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 12\cdot 4+2 = 50$$

$$\Rightarrow \nabla^2 f(\bar{x}) = \begin{pmatrix} 50 & +4 \\ +4 & -8 \end{pmatrix} \Rightarrow$$

$$\Rightarrow -\nabla^2 f(\bar{x})^{-1} = -\frac{1}{384}\begin{pmatrix} 8 & 4 \\ 4 & 50 \end{pmatrix}$$

$$\Rightarrow -\frac{1}{384}\begin{pmatrix} 8 & 4 \\ 4 & 50 \end{pmatrix}\begin{pmatrix} -44 \\ 24 \end{pmatrix} = \dots = \frac{2}{3}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

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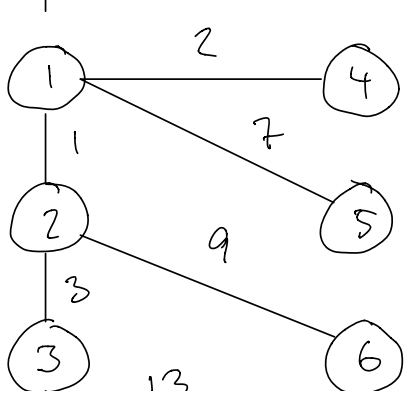
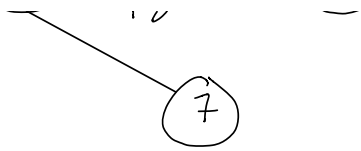
nod	2	3	4	5	6	7
1	1	6	2	7	16	14
2		3	4	11	9	15
3			5	9	12	13
4				10	17	18
5					21	19
6						20

a)





$1+2+7+9+3+12 = \underline{\underline{35}}$

b) Låt $x_{ij} = \begin{cases} 1, & \text{om bäge används} \\ 0, & \text{annars} \end{cases}$

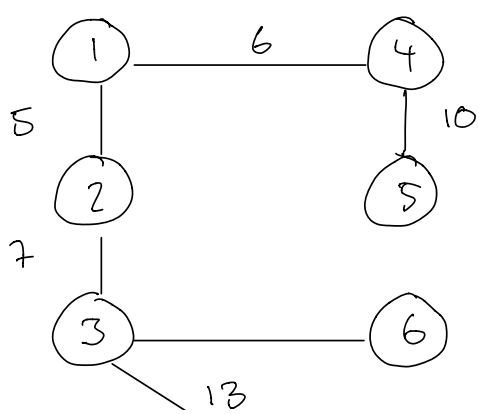
$$\min z = f(x)$$

då $\underbrace{x_{12}+x_{13}+x_{14}+x_{15}+x_{23}+x_{24}+x_{26}+x_{34}+x_{35}}_{g(x)} \leq 3$

$x_{ij} = 0/1 \quad , \quad i,j = 1,\dots,7$

$$h(v) = -3v + \min\big(x_{12}(1+v) + x_{13}(6+v) + x_{14}(2+v) + x_{15}(7+v)$$
$$+ x_{16}\cdot 16 + x_{17}\cdot 14 + x_{23}(3+v) + x_{24}(4+v) + x_{25}\cdot 11 + x_{26}(9+v) +$$
$$+ x_{27}\cdot 15 + x_{34}(5+v) + x_{35}(9+v) + x_{36}\cdot 12 + x_{37}\cdot 13 + x_{45}\cdot 10 + x_{46}\cdot 17$$
$$+ x_{47}\cdot 18 + x_{56}\cdot 21 + x_{57}\cdot 19 + x_{67}\cdot 20$$

$$h(4) = -12 + \min\big(5x_{12} + 10x_{13} + 6x_{14} + 11x_{15} + 16x_{16} + 14\cdot x_{17} +$$
$$+ 7x_{23} + 8x_{24} + 11x_{25} + 13x_{26} + 15x_{27} + 9x_{34} + 13x_{35} + 12x_{36} +$$
$$+ 13x_{37} + 10x_{45} + 17x_{46} + x_{47}\cdot 18 + x_{56}\cdot 21 + x_{57}\cdot 19 + x_{67}\cdot 20 \big)$$



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$h(y) = -12 + 5 + 6 + 7 + 12 + 13 + 10 = \underline{\underline{41}}$ Tillåten ty

$\Rightarrow \quad z^* \leq 1 + 2 + 3 + 12 + 13 + 10 = 41$

$z^* \geq 41$

$\Rightarrow$ Skattningen sammanfaller $\Rightarrow$ kostnad 41 optimalt.